Lecture 1 - Objective and Outcomes

You will often see $(\Omega, \mathcal{F}, P)$ used to denote a probability space. This lecture explains where this comes from and how it relates to reality.

- Experiments, Outcomes ω, Sample space Ω.

- Collections of events $\mathcal{F}$.

- Probability measure P.

- Counting outcomes.
- Bayes' theorem.
- Independence.

After reviewing the notes you should:

- know the distinction between events and outcomes,
- understand how probability measures correspond to our intuitive ideas about how likely events are,
- be able to work out the appropriate method for counting outcomes for any given situation,
- understand how conditioning can be reversed using Bayes' theorem,

- know a large number of the consequences of events being independent

Sample space for throwing die twice

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Definition: collection of events, $\mathcal{F}$

For any experiment, the events form a collection of subsets of Ω which we denote $\mathcal{F}$. The collection $\mathcal{F}$ has the following properties:

i. $\emptyset \in \mathcal{F}$,

ii. if $A \in \mathcal{F}$ then$A^c \in \mathcal{F}$,

iii. if $A_1, A_2, \ldots, \in \mathcal{F}$then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition: probability measure, P

P on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \longrightarrow [0, 1]$ satisfying;

i. $P(A) \geq 0$,

ii. $P(\Omega) = 1$,

iii. if $A_1, A_2, \ldots,$ is a collection of mutually exclusive members of $\mathcal{F}$ then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Properties of probability measures

a. $P(A^c) = 1 - P(A)$.

b. If $A \subseteq B$ then
$P(B) = P(A) + P(B \backslash A) \geq P(A)$.

c. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

d. More generally if $A_1, \ldots, A_n$ are events then
$P(\bigcup_{i=1}^n A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j)$
$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n+1} P(A_1 \cap \ldots \cap A_n)$.

e. Boole inequality: $P(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$.

Examples taken from Larsen and Marx,
An Introduction to Mathematical Statistics

Birthday problems: What is the smallest number of people needed to have probability of over 0.5 of:

a. at least one person with the same birthday as me?
b. at least two people with the same birthday as each other?

Poker problems

a poker hand is five cards dealt from a (perfectly shuffled) standard 52 card pack. What is the probability of:

a. a full house?

b. a pair?

c. a straight?

Partitions

A set of events $B_1, \ldots, B_n$ defined on Ω is a partition if:

1. Exhaustive: $\bigcup_{i=1}^{n} B_i = \Omega$,
2. Mutually exclusive: $B_i \cap B_j = \emptyset$ for $i \neq j$,
3. Non-zero probability: $P(B_i) > 0$
   for $i = 1, \ldots, n$.

Some properties of independent events

1. If $P(A) > 0$ then $P(B|A) = P(B) \iff A \perp B$.
   If $P(B) > 0$ then $P(A|B) = P(A) \iff A \perp B$.

2. If $A \perp B$ then $A^c \perp B^c$, $A^c \perp B$ and $A \perp B^c$.